Finite-dimensional approximations of the resolvent of an infinite band matrix and continued fractions

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Abstract

In this paper we study the approximability of the resolvent of an operator generated by a band matrix by means of the resolvents of its finite-dimentional sections. For bounded perturbations of selfadjoint matrices a positive result in a large domain is obtained. We apply it to tridiagonal complex matrices in order to establish convergence conditions for Chebyshev continued fraction on sets of the complex domain. In the particular case of compact perturbation, this result is sharpened and the connection between the poles of the limit function and the eigenvalues of the tridiagonal matrix is shown.

§1. Introduction and statement of the main result

As it is well known, the properties of a continued fraction can be studied via the operator generated by the tridiagonal (Jacobi) matrix constructed with the coefficients of the continued fraction. This method has been widely used, and shortness of the proofs is a strong argument for that this is an adecuate approach.

In the setting of real and symmetric tridiagonal matrix the main tool of investigation are the classical moment problem, the theory of general orthogonal polynomials and the spectral theory of selfadjoint operators (see e. gr. [1], [13]).

If we regard continued fractions with complex coefficients or study the Hermite-Padé approximants (to mention only some examples), non-Hermitian or non-tridiagonal band matrices arise. Then, generally we can not use the spectral theorem, neither orthogonal polynomials, and other techniques must be applied.

In this article, that consists of four Sections, we aim at some extension of the classical theory considering the case of a bounded perturbation of an unbounded selfadjoint operator, when both are generated by band matrices (Sections 1 and 2). Then we apply the main result to a class of Chebyshev continued fractions in order to establish convergence conditions on sets of the complex domain. In the Section 4, where we study the particular

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case of compact perturbations, this result is sharpened and the connection between the poles of the limit function and the eigenvalues of the tridiagonal matrix is shown.

In this paper we are mainly interested in continued fractions. Some other applications of the main result (Theorem 1) will appear elsewhere.

In what remains of this Section, we introduce some notation and formulate the results concerning convergence of operators.

We consider infinite band or $(2p+1)$-diagonal matrices

$$G = (g_{ij})_{i,j=0}^{\infty} \tag{1}$$

with $g_{ij} = 0$ for $|i-j| > p$. For every sequence $x = (x_0, x_1, \ldots)$ of complex numbers denote by Gx the sequence with components $(Gx)_n$ given by the formal multiplication of G by the vector x.

To pass from the matrix G to operators we introduce the following linear sets in the Hilbert space l^2: D_0, consisting of vectors with finitely many nonzero components, and $D(G) = \{x \in l^2 : Gx \in l^2\}$; both sets are dense in l^2, and provided G is a band matrix, $D_0 \subset D(G)$ (see [8]). We define the operator $\mathcal{G}$ on $D(G)$ by the equality $\mathcal{G}(x) = Gx$ and denote by $\mathcal{G}_0$ the restriction of $\mathcal{G}$ to D_0. We call the operators $\mathcal{G}_0$ and $\mathcal{G}$, respectively, the minimal and maximal operators generated by the matrix G (for a slightly different definition see e.gr. [2]). Moreover, in the sequel we shall denote the matrix and the maximal operator $\mathcal{G}$ by the same letter G whenever this cannot lead us into confusion.

A band matrix G generates a bounded operator in l^2 if and only if all its entries g_{ij} are uniformly bounded; in this case the above matrix representation is valid for all $x \in l^2$ and we call such a matrix *bounded*. In particular, if a bounded matrix G is hermitian (i.e. $g_{ij} = \bar{g}_{ji}$), then $\mathcal{G}$ is a selfadjoint operator.

In the general case of an hermitian matrix G at least we can assure that $\mathcal{G}_0$ is symmetric, and in consequence, admits a clausure that we denote by $\mathcal{G}_0'$. It is not difficult to establish that the operator $\mathcal{G}$ is the adjoint of $\mathcal{G}_0'$: $\mathcal{G} = [\mathcal{G}_0']^*$. If $\mathcal{G}$ is selfadjoint (that means that $\mathcal{G} = \mathcal{G}_0'$) then we say that matrix G is *selfadjoint*.

Let $\{e_i\}_{i=0}^{\infty}$, $e_i = (0, 0, \ldots, 0, \overbrace{1}^{(i+1)}, 0, \ldots)^T$, be the standard basis of l^2. For each fixed $n \in \mathbf{N}$, we consider the operators $E_n : \ell^2 \longrightarrow \mathbf{C}^n$ such that $E_n x = (x_1, x_2, \ldots, x_n)$ for each $x = (x_1, x_2, \ldots) \in \ell^2$ and $\widetilde{E}_n : \mathbf{C}^n \longrightarrow \ell^2$ such that $\widetilde{E}_n(x_1, x_2, \ldots x_n) = (x_1, x_2, \ldots, x_n, 0, \ldots) \in \ell^2$. It is clear that the matrix representation of $\widetilde{E}_n E_n$ in the basis $\{e_i\}$ is given by the block matrix

$$\widetilde{E}_n E_n = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}_{\infty \times \infty} \quad \text{with} \quad I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}_{n \times n},$$

and that $E_n \widetilde{E}_n = I_n : \mathbf{C}^n \longrightarrow \mathbf{C}^n$.

For an infinite matrix $G = (g_{ij})_{i,j=0}^{\infty}$ we set the finite-dimentional section $G_n = (g_{ij})_{i,j=0}^{n-1}$; in other words, G_n is the matrix of order $n \times n$ defined by the first n rows and columns of G. Each $\widetilde{E}_n G_n E_n$ generates a bounded operator on l^2 with a finite-dimentional range, whose matrix representation is

$$\begin{pmatrix} G_n & 0 \\ 0 & 0 \end{pmatrix}_{\infty \times \infty}.$$

In the sequel we identify the spectrum of a band matrix G with the spectrum of the maximal operator generated by G. Hence, the spectrum $\sigma(G)$ is the set of all $z \in \mathbf{C}$ such that $(G - zI)^{-1}$ is not a bounded linear operator on l^2, $\sigma_p(G) \subset \sigma(G)$ is the point spectrum (set of eigenvalues) of G, and $\rho(G) = \mathbf{C} \setminus \sigma(G)$ is the resolvent set of G.

The resolvent operator

$$\mathcal{R}^{(G)}(z) = (G - zI)^{-1}$$

is defined and bounded on l^2 for $z \in \rho(G)$. Moreover, if $z \notin \sigma(G_n)$, then $(G_n - zI_n)^{-1}$ is one-to-one on $\mathbf{C}^n$, and the operator

$$\mathcal{R}_n^{(G)}(z) = \widetilde{E}_n (G_n - zI_n)^{-1} E_n$$

is bounded on ℓ^2.

Let

$$\mathcal{P}(G) = \bigcap_{n \geq 0} \overline{\bigcup_{k \geq n} \sigma(G_k)}\,.$$

Clearly, $\mathcal{P}(G)$ can be characterized as the set of all $z \in \mathbf{C}$ such that there exists a sequence $\{z_n\}_{n \in \Lambda}$, $z_n \in \sigma(G_n)$, $\forall n \in \Lambda \subset \mathbf{N}$, satisfying

$$\lim_{n \in \Lambda} z_n = z\,.$$

Finally, by $\operatorname{dist}(z, A)$ we understand the usual Euclidean distance on $\mathbf{C}$ from the point z to the set A, $\|\cdot\|$ denotes the usual norm in ℓ^2, $\|\cdot\|_n$ denotes the Euclidean norm in $\mathbf{C}^n$, and $\longrightarrow$ referring to operators means the strong (or ordinary) convergence.

All our further considerations concern the case when the matrix G can be represented in the form

$$G = H + C\,, \tag{2}$$

where the band matrices H and C are respectively selfadjoint and bounded. Note a certain freedom in the selection of matrices H and C.

Theorem 1 *If* (2) *holds then*

$$\mathcal{R}_n^{(G)}(z) \longrightarrow \mathcal{R}^{(G)}(z) \tag{3}$$

uniformly on compact subsets of $\{z : \operatorname{dist}(z, \mathcal{P}(H) \cup \sigma(H)) > \|C\|\}$.

In particular, since all $\sigma(H_n) \subset \mathbf{R}$, $\sigma(H) \subset \mathbf{R}$, we can state the following

Corollary 1 *If* (2) *holds then* (3) *takes place uniformly on compact subsets of* $\{z : |\mathrm{Im}\ z| > \|C\|\}$.

Moreover, for tridiagonal matrices a more precise result can be derived. In fact, it is well known that in this case all $\sigma(H_n)$ lie in the convex hull $conv(\sigma(H))$ of $\sigma(H)$, so that we have

Corollary 2 *If H is tridiagonal and* (2) *holds,* (3) *takes place uniformly on any compact set K such that* $\operatorname{dist}(K, \operatorname{conv}(\sigma(H))) > \|C\|$.

If S is the right shift in l^2 generated by the infinite matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{\infty\times\infty}$$

then we denote $G^{(k)} = [S^k]^T G S^k$. In other words, matrix $G^{(k)}$ is obtained from G by deleting its first k rows and columns. By proving that $G^{(k)}$ conserves the structure given by (2) we have

Corollary 3 *If* (2) *holds then*

$$\mathcal{R}_n^{(G^{(k)})}(z) \longrightarrow \mathcal{R}^{(G^{(k)})}(z) \ . \tag{4}$$

uniformly on compact subsets of $\{z : \mathrm{dist}\left(z, \mathcal{P}(H^{(k)}) \cup \sigma(H^{(k)})\right) > \|C^{(k)}\|\}$.

Finally, taking $H = 0$, we have the following

Corollary 4 *If* $\|G\| < \infty$ *then* (3) *takes place uniformly on compact sets of* $\{|z| > \|G\|\}$.

§2. Proof of Theorem 1

We divide the proof of the main result into several lemmas that may present an independent interest.

Lemma 1 *If* $\|C\| < \infty$ *then* $\widetilde{E}_n C_n E_n \longrightarrow C$.

Proof: Since $\widetilde{E}_n E_n$ is the projector of ℓ^2 onto $span\{e_0, \dots, e_{n-1}\}$ and $C_n = E_n C \widetilde{E}_n$, we have $\widetilde{E}_n C_n E_n = \widetilde{E}_n E_n C \widetilde{E}_n E_n$. Then $\|\widetilde{E}_n C_n E_n\| \le \|C\|$. On the other hand,

$$(C - \widetilde{E}_n C_n E_n) e_j = (I - \widetilde{E}_n E_n)\vec{c}_j \longrightarrow 0, \ n \to \infty \ ,$$

where $\vec{c}_j$ is the j-th column of the matrix C. Hence, $\|\widetilde{E}_n C_n E_n\|$ are uniformly bounded and we have convergence in a dense subset of l^2. ■

Lemma 2 *If* $\mathrm{dist}(z, \sigma(H)) > \|C\|$ *then* $z \in \rho(G)$. *Moreover,*

$$\|\mathcal{R}^{(G)}(z)\| \le \frac{1}{\mathrm{dist}(z, \sigma(H)) - \|C\|} \ . \tag{5}$$

Proof: If $\|G\| < \infty$ (or what is the same, $\|H\| < \infty$) then this is a simple consequence of the well-known theorem on the invertibility of a small perturbation of a bounded invertible operator. In the general case we apply the following identity

$$\mathcal{R}^{(G)}(z) = \mathcal{R}^{(H)}(z) \left[I + C\mathcal{R}^{(H)}(z) \right]^{-1} \ , \tag{6}$$

that is valid whenever it makes sense. Moreover, we use that for a selfadjoint operator H and $z \in \rho(H)$,

$$\|\mathcal{R}^{(H)}(z)\| = \frac{1}{\operatorname{dist}(z, \sigma(H))} \tag{7}$$

(see e. gr. [11, Problem III–6.16]). Hence, if $\operatorname{dist}(z, \sigma(H)) > \|C\|$,

$$\|C\mathcal{R}^{(H)}(z)\| \leq \|C\| \frac{1}{\operatorname{dist}(z, \sigma(H))} < 1\,,$$

operator $\left[I + C\mathcal{R}^{(H)}(z)\right]^{-1}$ exists and is bounded, so that $\mathcal{R}^{(G)}(z)$ is a product of bounded operators.

Now, the identity

$$\mathcal{R}^{(H)}(z) - \mathcal{R}^{(G)}(z) = \mathcal{R}^{(H)}(z) C \mathcal{R}^{(G)}(z) \tag{8}$$

implies that if $z \in \rho(H) \cap \rho(G)$,

$$\|\mathcal{R}^{(G)}(z)\| \leq \|\mathcal{R}^{(H)}(z)\| + \|\mathcal{R}^{(H)}(z)\| \|C\| \|\mathcal{R}^{(G)}(z)\|$$

so that

$$\|\mathcal{R}^{(G)}(z)\| \left(1 - \|C\| \|\mathcal{R}^{(H)}(z)\|\right) \leq \|\mathcal{R}^{(H)}(z)\|$$

and it remains to apply (7). ∎

In the following step we prove that the norms of $\mathcal{R}_n^{(G)}(z)$ are uniformly bounded. In fact, we have

Lemma 3 *For n sufficiently large, $\|\mathcal{R}_n^{(G)}(z)\|$ is uniformly bounded on compact subsets of $\{z \,:\, \operatorname{dist}(z, \mathcal{P}(H)) > \|C\|\}$.*

Proof: We have that $\|\mathcal{R}_n^{(G)}(z)\| \leq \|(G_n - zI_n)^{-1}\|_n$; then going through the proof of Lemma 2 we can obtain that for $\operatorname{dist}(z, \sigma(H_n)) > \|C_n\|_n$,

$$\|\mathcal{R}_n^{(G)}(z)\| \leq \frac{1}{\operatorname{dist}(z, \sigma(H_n)) - \|C_n\|_n}\,.$$

By Lemma 1, $\|C_n\|_n = \|\widetilde{E}_n C_n E_n\| \to \|C\|$. On the other hand, given an arbitrary $\varepsilon > 0$, for n sufficiently large

$$\operatorname{dist}(z, \sigma(H_n)) \geq \operatorname{dist}(z, \mathcal{P}(H)) - \varepsilon\,.$$

Hence, we can assure that, say

$$\|\mathcal{R}_n^{(G)}(z)\| \leq \frac{2}{\operatorname{dist}(z, \mathcal{P}(H)) - \|C\|}\,, \quad n \geq n_0(z)\,.$$

On the other hand, from the resolvent equation

$$\mathcal{R}^{(G)}(z) - \mathcal{R}^{(G)}(z_0) = (z - z_0)\mathcal{R}^{(G)}(z)\mathcal{R}^{(G)}(z_0)\,, \quad z, z_0 \in \rho(G)\,, \tag{9}$$

we have that

$$\|\mathcal{R}^{(G)}(z)\| \leq \frac{\|\mathcal{R}^{(G)}(z_0)\|}{1 - |z - z_0| \|\mathcal{R}^{(G)}(z_0)\|}\,, \quad |z - z_0| < \frac{1}{\|\mathcal{R}^{(G)}(z_0)\|}.$$

¿From this we obtain that $\mathcal{R}^{(G)}(z)$ is uniformly bounded on disks

$$\{z : |z - z_0| \le \delta(z_0)\}, \quad \delta(z_0) < \frac{1}{\|\mathcal{R}^{(G)}(z_0)\|}$$

for each $z_0 \in \{z : \operatorname{dist}(z, \mathcal{P}(H)) > \|C\|\}$, and it remains to apply standard compacity arguments to establish the uniform boundeness on the above mentioned compact sets. ∎

Now we deduce convergence in the selfadjoint case. As it is shown in the next Section, the following Lemma is a generalized form of the classical Stieltjes' theorem on the convergence of continued fractions.

Lemma 4

$$\mathcal{R}_n^{(H)}(z) \longrightarrow \mathcal{R}^{(H)}(z)$$

uniformly on each compact subset of $\rho(H) \setminus \mathcal{P}(H)$.

Proof: Define

$$v_j = (H - zI)e_j\,, \quad j = 0, 1, \ldots$$

Since $z \in \rho(H)$,

$$\mathcal{R}^{(H)}(z)v_j = e_j\,.$$

In order to prove that $span\{v_j,\ j = 0, 1, \ldots\}$ is dense in l^2, it is sufficient to establish that

$$< x, v_j >= 0\,, \quad x \in l^2\,, \quad j = 0, 1, \ldots \tag{10}$$

if and only if $x = 0$.

In fact, for every vector x both sides of the following equation exist and verify

$$< x, (H - zI)e_j > = < (H - \bar{z}I)x, e_j >\,, \quad j = 0, 1, \ldots\,.$$

Hence, if (10) holds,

$$< (H - \bar{z}I)x, e_j >= 0\,, \quad j = 0, 1, \ldots\,,$$

so that

$$(H - \bar{z}I)x = 0\,. \tag{11}$$

Note that, generally speaking, not every vector $(H - \bar{z}I)x$ must be in l^2, but if (11) holds, it means that $Hx = \bar{z}x$, so that $Hx \in l^2$ and (recall the definition of $D(H)$) $x \in D(H)$. But $\mathcal{P}(H) \cup \sigma(H) \subset \mathbf{R}$, hence $\bar{z} \notin \sigma_p(H)$ and we have that $x = 0$.

On the other hand we have

$$E_n v_j = E_n(H - zI)e_j = (H_n - zI_n)E_n e_j \in \mathbf{C}^n\,, \forall n \ge n_0(j)\,, \forall j = 0, 1, \ldots\,.$$

Then,

$$\mathcal{R}_n^{(H)}(z)v_j = \widetilde{E}_n(H_n - zI_n)^{-1}E_n v_j = \widetilde{E}_n E_n e_j = e_j = \mathcal{R}^{(H)}(z)v_j\,, j = 0, 1, \ldots$$

for n sufficiently large (actually, for $n > j + p$, if we suppose H $(2p + 1)$-diagonal).

In this way, we have established that

$$\left[\mathcal{R}^{(H)}(z) - \mathcal{R}_n^{(H)}(z)\right] x \longrightarrow 0\,, \quad n \to \infty$$

for x in a dense subset of l^2. Setting $C=0$ in Lemma 3 we have that $\|\mathcal{R}_n^{(H)}(z)\|$ are uniformly bounded on each compact subset of $\mathbf{C}\setminus\mathcal{P}(H)=\rho(H)\setminus\mathcal{P}(H)$, and the convergence on each $z\in\mathbf{C}\setminus\rho(H)$ follows. Taking into account this fact, the inequality

$$\begin{aligned}\|\mathcal{R}_n^{(H)}(z)x-\mathcal{R}^{(H)}(z)x\| &\le \|\mathcal{R}_n^{(H)}(z)x-\mathcal{R}_n^{(H)}(z_0)x\|+\|\mathcal{R}_n^{(H)}(z_0)x-\mathcal{R}^{(H)}(z_0)x\|\\ &+ \|\mathcal{R}^{(H)}(z_0)x-\mathcal{R}^{(H)}(z)x\|,\end{aligned}$$

and the resolvent equation (9) (for H and H_n) the assertion of this Lemma readily follows. ■

Now we will need one more tool: in order to ensure convergence of approximate inverses of some matrices we prove a version of Kantorovich's theorem (see e.gr. [10, Ch. 14]) that we state in the following weak form, sufficient for our purposes.

Lemma 5 *Suppose A is a bounded invertible operator on l^2; consider a sequence of invertible operators $A_n:\mathbf{C}^n\to\mathbf{C}^n$ such that*

$$\widetilde{E}_nA_nE_n\longrightarrow A\,, \tag{12}$$

and

$$\|A_n^{-1}\|_n\le K \tag{13}$$

(K independent of n). Then

$$\widetilde{E}_nA_n^{-1}E_n\longrightarrow A^{-1}\,.$$

Proof: We follow the scheme given in the paper [12, Th. 4.1]. For any $y\in l^2$ take $x=A^{-1}y$. Then, since $E_n\widetilde{E}_n=I_n:\mathbf{C}^n\to\mathbf{C}^n$,

$$\begin{aligned}\widetilde{E}_nA_n^{-1}E_ny-A^{-1}y &= \widetilde{E}_nA_n^{-1}E_n(Ax-\widetilde{E}_nA_nE_nx)+(\widetilde{E}_nA_n^{-1}E_n\widetilde{E}_nA_nE_nx-x)=\\ &= \widetilde{E}_nA_n^{-1}E_n(Ax-\widetilde{E}_nA_nE_nx)+(\widetilde{E}_nE_nx-x)\end{aligned}$$

Obviously, $(\widetilde{E}_nE_n-I)x\to 0$, and by (12) and (13) the first term also converges towards zero. ■

Now we are ready to prove the main result of this Section.
For z satisfying the hypotesis, by Lemma 4

$$\mathcal{R}_n^{(H)}(z)\longrightarrow\mathcal{R}^{(H)}(z)\,,$$

so that using Lemma 1 we have that

$$\widetilde{E}_nC_nE_n\mathcal{R}_n^{(H)}(z)\longrightarrow C\mathcal{R}^{(H)}(z)\,. \tag{14}$$

Hence, from (7),

$$\|C_n(H_n-zI_n)^{-1}\|_n\le\|C_n\|_n\|(H_n-zI_n)^{-1}\|_n\to\frac{\|C\|}{\mathrm{dist}(z,\sigma(H))}<1\,,$$

that means that

$$\|C_n(H_n-zI_n)^{-1}\|_n\le M<1\,,\quad n\ge n_0(z)\,.$$

In consequence, there exist bounded inverses of operators $I_n+C_n(H_n-zI_n)^{-1} : \mathbf{C}^n \to \mathbf{C}^n$ and

$$\| \left[I_n + C_n(H_n - zI_n)^{-1} \right]^{-1} \|_n \leq \frac{1}{1-M} .$$

Since on the other hand, by (14)

$$\widetilde{E}_n \left[I_n + C_n(H_n - zI_n)^{-1} \right] E_n = \widetilde{E}_n E_n + (\widetilde{E}_n C_n E_n)\mathcal{R}_n^{(H)}(z) \longrightarrow I + C\mathcal{R}^{(H)}(z),$$

and we can apply Lemma 5. In this way we obtain that

$$\widetilde{E}_n \left[I_n + C_n(H_n - zI_n)^{-1} \right]^{-1} E_n \longrightarrow \left[I + C\mathcal{R}^{(H)}(z) \right]^{-1} .$$

It remains to apply the identity

$$\mathcal{R}_n^{(G)}(z) = \mathcal{R}_n^{(H)}(z)\widetilde{E}_n \left[I_n + C_n(H_n - zI_n)^{-1} \right]^{-1} E_n. \tag{15}$$

Finally, being all the resolvents (operator valued) analytic and uniformly bounded functions on compact subsets of the specified domain, uniform convergence holds. Theorem 1 is proved. ■

In the previous Section we stated without proof some consequences of Theorem 1. Corollaries 1, 2 and 4 are immediate; for Corollary 3 we just need to establish the following auxiliary result.

Lemma 6 *If H is selfadjoint then $H^{(k)}$ is also selfadjoint for all $k \in \mathbf{N}$.*

Proof: Clearly, it is sufficient to verify the statement for $k = 1$. We have to show that the maximal operator $H^{(1)}$ is symmetric, i.e.

$$< H^{(1)}u^{(1)}, v^{(1)} >=< u^{(1)}, H^{(1)}v^{(1)} > , \quad \forall u^{(1)} , v^{(1)} \in D(H^{(1)})$$

But for

$$u^{(1)} = (u_1^{(1)}, u_2^{(1)}, \ldots), \quad v^{(1)} = (v_1^{(1)}, v_2^{(1)}, \ldots) \in D(H^{(1)})$$

given, vectors $u = (0, u_1^{(1)}, u_2^{(1)}, \ldots)$ and $v = (0, v_1^{(1)}, v_2^{(1)}, \ldots)$ lie in $D(H)$, and equation

$$< H^{(1)}u^{(1)}, v^{(1)} >=< Hu, v >=< u, Hv >=< u^{(1)}, H^{(1)}v^{(1)} >$$

holds, that completes the proof. ■

Remark 1 Going through the proofs given above it is clear that the same results hold for "sparse" matrices, that is for the matrices having a finite number of non-zero elements on each row and column.

§3. Tridiagonal matrices and continued fractions

We study in particular the tridiagonal matrices due to their connection with continued fractions. In fact, consider an infinite Tchebyshev continued fraction (J-fraction)

$$f(z) = \cfrac{1}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \cfrac{a_2^2}{z - b_2 - \ddots}}}\,, \tag{16}$$

with the natural condition

$$a_n \neq 0\,, n = 1, 2, \dots\,, \tag{17}$$

and let for each $n \in \mathbf{N}$

$$f_n(z) = \frac{Q_n(z)}{P_n(z)} = \cfrac{1}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \ddots \cfrac{a_{n-1}^2}{z - b_{n-1}}}} \tag{18}$$

be its n-th convergent. Under (17), $\deg P_n = n$ and we take $P_n(z)$ monic. It is widely known and easy to verify that the polynomials P_n and Q_n satisfy three-term recurrence relations,

$$\begin{aligned} P_{n+1}(z) = (z - b_n)P_n(z) - a_n^2 P_{n-1}(z), \quad n \geq 1, \\ P_0(z) = 1,\; P_1(z) = z - b_0, \end{aligned} \tag{19}$$

and

$$\begin{aligned} Q_{n+1}(z) = (z - b_n)Q_n(z) - a_n^2 Q_{n-1}(z), \quad n \geq 1, \\ Q_0(z) = 0, \quad Q_1(z) = 1. \end{aligned} \tag{20}$$

This gives us the connection with the tridiagonal matrix

$$G = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \dots \\ 0 & a_2 & b_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{21}$$

In fact, using (19) it can be easily established that

$$P_n(z) = (-1)^n \det(G_n - zI_n)\,,\; n \in \mathbf{N}\,.$$

In other words, zeros of P_n given in (19) coincide with the spectrum $\sigma(G_n)$.

In the sequel we shall make use of analogous polynomials connected with the shifted matrix $G^{(k)}$. Put

$$P_n^{(k)}(z) = (-1)^n \det(G_n^{(k)} - zI_n)\,, \quad n \in \mathbf{N},$$

(in particular, $P_n^{(0)}(z) = P_n(z)$); they verify the recurrence relation

$$\begin{aligned} P_{n+1}^{(k)}(z) = (z - b_{n+k})P_n^{(k)}(z) - a_{n+k}^2 P_{n-1}^{(k)}(z), \quad n \geq 1, \\ P_0^{(k)}(z) = 1,\; P_1^{(k)}(z) = z - b_k. \end{aligned} \tag{22}$$

A useful formula relating polynomials with different shifts is obtained expanding $\det(G_n^{(k)} - zI_n)$ along its first row:

$$P_n^{(k)}(z) = (z - b_k)P_{n-1}^{(k+1)}(z) - a_{k+1}^2 P_{n-2}^{(k+2)}(z)\,,\; k \geq 0\,,\; n \geq 1. \tag{23}$$

Some formulas below will look simpler with a different normalization of $P_n^{(k)}(z)$. Define

$$p_n^{(k)}(z) = \frac{P_n^{(k)}(z)}{a_k a_{k+1} \cdots a_{k+n}}\,, \quad \forall n \geq 0\,, \tag{24}$$

assuming $a_0 = 1$. Then (22) can be rewritten in the following way

$$\begin{aligned} a_{n+k+1}p_{n+1}^{(k)}(z) = (z - b_{n+k})p_n^{(k)}(z) - a_{n+k}p_{n-1}^{(k)}(z), \quad n \geq 1, \\ p_0^{(k)}(z) = \tfrac{1}{a_k},\; p_1^{(k)}(z) = \tfrac{z-b_k}{a_k a_{k+1}} \end{aligned} \tag{25}$$

and (23) takes the form

$$a_k p_n^{(k)}(z) = (z - b_k)p_{n-1}^{(k+1)}(z) - a_{k+1}p_{n-2}^{(k+2)}(z). \tag{26}$$

(For another proof of this equation see, e. gr. [14]).

¿From (22) and (20) it is clear that $Q_n \equiv P_{n-1}^{(1)}$, and we have (see (18))

$$f_n(z) = \frac{P_{n-1}^{(1)}(z)}{P_n(z)} = \frac{p_{n-1}^{(1)}(z)}{p_n(z)}\,, \tag{27}$$

where $p_n(z) = p_n^{(0)}(z)$.

The cornerstone of the convergence analysis of the continued fraction (16) in terms of operators lies in the following known expression for the convergents (see, e. gr. [6, Theorem 3.7, p. 131]):

$$f_n(z) = - < \mathcal{R}_n^{(G)}(z)e_0, e_0 >\,, \quad n \in \mathbf{N}\,, \quad z \in \mathbf{C} \setminus \sigma(G_n). \tag{28}$$

Hence, even the weak convergence of resolvents $\mathcal{R}_n^{(G)}(z)$ is sufficient for convergence of $f_n(z)$. It is clear now that Theorem 1, established in Section 2, has the following immediate consequence:

Theorem 2 *a) If matrix G given in* (21) *admits a representation* (2)*, then $f_n(z)$ converge uniformly at least on compact subsets of $\{z \in \mathbf{C} : \mathrm{dist}(z, \mathrm{conv}\,(\sigma(H))) > \|C\|\}$ to the analytic function*

$$f(z) = - < \mathcal{R}^{(G)}(z)e_0, e_0 > .$$

b) For any continued fraction (16) *with uniformly bounded complex coefficients,*

$$f_n(z) \underset{n}{\rightrightarrows} f(z) \tag{29}$$

at least on compact sets of $\{|z| > \|G\|\}$.

Remark 2 We should point out that the second statement is in some sense a refinement of the Worpitzky's Theorem on convergence of continued fractions with complex coefficients (see e.gr. [15, Ch. III]).

Remark 3 If in (2)

$$H = \begin{pmatrix} \beta_0 & \alpha_1 & 0 & \cdots \\ \alpha_1 & \beta_1 & \alpha_2 & \cdots \\ 0 & \alpha_2 & \beta_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

then (29) holds on compact sets K that verify $\operatorname{dist}(K, [\gamma_-, \gamma_+]) > M$, where $\gamma_- = \liminf (\beta_n - |\alpha_n| - |\alpha_{n+1}|)$, $\gamma_+ = \limsup (\beta_n + |\alpha_n| + |\alpha_{n+1}|)$ and $M = \sup (|b_n - \beta_n| + |a_n - \alpha_n| + |a_{n+1} - \alpha_{n+1}|)$. These bounds readily follow from the Gershgorin Theorem and estimations of the operator norm of a matrix (see e.gr. [16, pp. 71, 72]).

Remark 4 A simple generalization of formula (28) can be obtained. In fact, solving a corresponding linear system we have

$$p_{i-1}(z)\frac{p_{n-j}^{(j)}(z)}{p_n(z)} = - < \mathcal{R}_n^{(G)}(z)e_{j-1}, e_{i-1} >, \quad n \in \mathbf{N}, \quad z \in \mathbf{C} \setminus \sigma(G_n).$$

Hence, Theorem 1 allows to establish a somehow stronger version of Theorem 2 that we state as

Corollary 5 *Under assumptions of a) of Theorem 2,*

$$p_{i-1}(z)\frac{p_{n-j}^{(j)}(z)}{p_n(z)} \underset{n}{\rightrightarrows} - < \mathcal{R}^{(G)}(z)e_{j-1}, e_{i-1} >$$

on compact subsets of $\{z \in \mathbb{C} : \operatorname{dist}(z, \operatorname{conv} \sigma(H)) > \|C\|\}$.

§4. Compact perturbations of Jacobi matrices

In this Section we improve the results on convergence of continued fractions, obtained above, but in the particular case of matrix $C = (c_{ij})$ tridiagonal and compact, that is equivalent to

$$c_{ij} = 0 \text{ if } |i - j| > 1, \quad \lim_n c_{n+k,n} = 0, \; k \in \mathbf{Z}. \tag{30}$$

We maintain the definitions and notation introduced above.

The key fact here resides in the limit

$$\lim_{k \longrightarrow \infty} \|C^{(k)}\| = 0 \tag{31}$$

that follows from (30). By it means we can "approach" the real axis and establish some more precise results even close to the convex hull of $\sigma(H)$. The connection between the resolvent operators of G and $G^{(k)}$ is proved using the following lemma that we state without proof since it can be immediately verified:

Lemma 7 *Let*

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \tag{32}$$

be an invertible finite matrix, where $A = A^T$, $C = C^T$, and A , C are invertible matrices. Then $(A - BC^{-1}B^T)$ and $(C - B^TA^{-1}B)$ are invertible matrices, and

$$M^{-1} = \begin{pmatrix} (A - BC^{-1}B^T)^{-1} & -A^{-1}B(C - B^TA^{-1}B)^{-1} \\ -CB^T(A - BC^{-1}B^T)^{-1} & (C - B^TA^{-1}B)^{-1} \end{pmatrix}. \qquad (33)$$

Additionally, we need a more precise description of the set $\mathcal{P}(G)$ in the case of compact perturbation, that was obtained in [4, Th. 2].

Lemma 8

$$\mathcal{P}(G) \subseteq \sigma_p(G) \cup \left[\bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)}))\right].$$

Now we are ready to state and prove the main result of this Section.

Theorem 3 *If (2) holds with C compact and tridiagonal, then*

$$\mathcal{R}_n^{(G)}(z) \longrightarrow \mathcal{R}^{(G)}(z)$$

uniformly on compact subsets of $\mathbf{C} \setminus \left[\sigma_p(G) \cup \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)}))\right]$.

Proof: We establish first that the sequence $\{\|\mathcal{R}_n^{(G)}(z)\|,\ n \geq n_0(z)\}$ is bounded for each $z \in \mathbf{C} \setminus [\sigma_p(G) \cup \bigcap_{k=0}^{\infty} conv(\sigma(H^{(k)}))]$. In order to simplify notation, in the sequel we identify finite-dimentional operators with their matrix representations.

Given $z \notin \sigma_p(G) \cup \bigcap_{k=0}^{\infty} conv(\sigma(H^{(k)}))$ we can find a $k_1 \in \mathbf{N}$ verifying

$$\operatorname{dist}(z, \mathcal{P}(H^{(k)}) \cup \sigma(H^{(k)})) \geq \operatorname{dist}(z, conv(\sigma(H^{(k_1)})))$$

for every $k \geq k_1$ (notice that $conv(\sigma(H^{(k)})) \subset conv(\sigma(H^{(k)}))$, $k \geq k_1$). Because

$$\operatorname{dist}(z, conv(\sigma(H^{(k_1)}))) > 0,$$

also we find a $\widetilde{k}_1 \in \mathbf{N}$ verifying

$$\operatorname{dist}(z, conv(\sigma(H^{(k_1)}))) > \|C^{(k)}\|$$

for every $k \geq \widetilde{k}_1$. Then there exists $k_0 = \max\{k_1, \widetilde{k}_1\}$ such that

$$\operatorname{dist}(z, \mathcal{P}(H^{(k)}) \cup conv(\sigma(H^{(k_1)}))) > \|C^{(k)}\|$$

for every $k \geq k_0$. Then, by Corollary 3, there exists $n_0 = n_0(z,k) \in \mathbf{N}$ such that both $G^{(k)} - zI$ and $G_n^{(k)} - zI_n$ are invertible for each $n \geq n_0$. Moreover, for these k, $\{(G_n^{(k)} - zI_n)^{-1}\}_{n \geq n_0}$ is bounded.

On the other hand, Lemma 8 establishes that

$$\exists n_1 = n_1(z)\ :\ G_n - zI_n \ \text{ is invertible } \ ,\forall n \geq n_1.$$

Fix $k \geq \max\{k_0, n_1\}$ and take $n \geq k+n_0$; then matrices $G_k - zI_k$, $G^{(k)}_{n-k} - zI_{n-k}$, $G_n - zI_n$ are invertible. Applying Lemma 7 to matrix $M = G_n - zI_n$ we obtain

$$(G_n - zI_n)^{-1} = \begin{pmatrix} D^{-1}_{n,k} & -(G_k - zI_k)^{-1} B_k F^{-1}_{n,k} \\ -(G^{(k)}_{n-k} - zI_{n-k})^{-1} B_k^T D^{-1}_{n,k} & F^{-1}_{n,k} \end{pmatrix}, \tag{34}$$

where

$$D_{n,k} = (G_k - zI_k) - B_k (G^{(k)}_{n-k} - zI_{n-k})^{-1} B_k^T$$

and

$$F_{n,k} = (G^{(k)}_{n-k} - zI_{n-k}) - B_k^T (G_k - zI_k)^{-1} B_k$$

are respectively $k \times k$ and $(n-k) \times (n-k)$ invertible matrices, and

$$B_k = a_k \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ 1 & \cdots & 0 \end{pmatrix} \tag{35}$$

is a finite matrix of order $k \times (n-k)$. Since the operator norm $\|B_k\| = |a_k|$ for all n, it is sufficient to establish that norms of the blocks $D^{-1}_{n,k}$ and $F^{-1}_{n,k}$ are uniformly bounded. We shall achieve it in two steps.
i) In order to find bounds for $D^{-1}_{n,k}$ observe that for

$$(G^{(k)}_{n-k} - zI_{n-k})^{-1} = \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n-k} \\ \vdots & & \vdots \\ \alpha_{n-k,1} & \cdots & \alpha_{n-k,n-k} \end{pmatrix} \tag{36}$$

it is immediate to verify that

$$B_k (G^{(k)}_{n-k} - zI_{n-k})^{-1} B_k^T = a_k^2 \alpha_{1,1} \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix}, \tag{37}$$

where $\alpha_{1,1} =< \mathcal{R}^{(G^{(k)})}_{n-k}(z) e_0, e_0 >= -f^{(k)}_{n-k}(z)$ (see (36) and (28)). Hence, $D_{n,k}$ is a tridiagonal matrix whose entries coincide with that of $G_k - zI_k$ except the last one on the main diagonal. In other words,

$$D_{n,k} = \begin{pmatrix} b_0 - z & a_1 & \ldots & 0 & 0 \\ a_1 & b_1 - z & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & b_{k-2} - z & a_{k-1} \\ 0 & 0 & \ldots & a_{k-1} & b_{k-1} - z + a_k^2 f^{(k)}_{n-k}(z) \end{pmatrix}. \tag{38}$$

In consequence, operators $D_{n,k} : \mathbf{C}^k \longrightarrow \mathbf{C}^k$ converge to $D_k : \mathbf{C}^k \longrightarrow \mathbf{C}^k$ given by

$$D_k = \begin{pmatrix} b_0 - z & a_1 & \ldots & 0 & 0 \\ a_1 & b_1 - z & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & b_{k-2} - z & a_{k-1} \\ 0 & 0 & \ldots & a_{k-1} & b_{k-1} - z + a_k^2 f^{(k)}(z) \end{pmatrix}. \tag{39}$$

Since we are dealing with bounded operators in $\mathbf{C}^k$ (k constant), we garantee the uniform boundeness just establishing that D_k is invertible, that is, its matrix is no singular. Indeed, taking into account (19) as in [4, Th. 3] we arrive at

$$\det(D_k) = (-1)^k [P_k(z) - a_k^2 P_{k-1} f^{(k)}(z)] \neq 0. \tag{40}$$

ii) In order to find bounds for blocks $F_{n,k}^{-1}$, we write

$$(G_k - zI_k)^{-1} = \begin{pmatrix} \gamma_{1,1} & \cdots & \gamma_{1,k} \\ \vdots & & \vdots \\ \gamma_{k,1} & \cdots & \gamma_{k,k} \end{pmatrix}, \tag{41}$$

and it is immediate to verify that

$$B_k^T (G_k - zI_k)^{-1} B_k = a_k^2 \gamma_{k,k} \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \tag{42}$$

is a diagonal matrix of order $n-k$.

Hence, as before, $F_{n,k}$ is a tridiagonal matrix whose entries except the first one coincide with the entries of $G_{n-k}^{(k)} - zI_{n-k}$, but now the dimention of each block varies with n. Some finer computations need to be done in this case.

Since

$$(G_{n-k}^{(k)} - zI_{n-k}) - F_{n,k} = B_k^T (G_k - zI_k)^{-1} B_k,$$

we have

$$F_{n,k}^{-1} - (G_{n-k}^{(k)} - zI_{n-k})^{-1} = (G_{n-k}^{(k)} - zI_{n-k})^{-1} B_k^T (G_k - zI_k)^{-1} B_k F_{n,k}^{-1}. \tag{43}$$

Then

$$\|F_{n,k}^{-1}\| \leq \|(G_{n-k}^{(k)} - zI_{n-k})^{-1}\| \left[1 + \|B_k^T (G_k - zI_k)^{-1} B_k F_{n,k}^{-1}\|\right].$$

Because $\{\|(G_{n-}^{(k)} - zI_{n-k})^{-1}\|, n \geq k + n_0\}$ is a bounded sequence we conclude that it is sufficient to establish that also the sequence $\{\|B_k^T (G_k - zI_k)^{-1} B_k F_{n,k}^{-1}\|, n \geq k + n_0\}$ is bounded.

On the other hand, writing

$$(G_{n-k}^{(k)} - zI_{n-k})^{-1} = \frac{1}{\det(G_{n-k}^{(k)} - zI_{n-k})} \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n-k} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n-k} \\ \vdots & \vdots & & \vdots \\ g_{n-k,1} & g_{n-k,2} & \cdots & g_{n-k,n-k} \end{pmatrix} \tag{44}$$

($g_{i,j} = \alpha_{i,j} \det(G_{n-k}^{(k)} - zI_{n-k})$, $i,j = 1,\ldots,n-k$), and taking into account the structure of $F_{n,k}$ we have

$$F_{n,k}^{-1} = \frac{1}{\det(F_{n,k})} \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n-k} \\ g_{2,1} & f_{2,2} & \cdots & f_{2,n-k} \\ \vdots & \vdots & & \vdots \\ g_{n-k,1} & f_{n-k,2} & \cdots & f_{n-k,n-k} \end{pmatrix}, \; f_{i,j} \in \mathbf{C}, \; i,j = 2,\ldots,n-k. \tag{45}$$

Futhermore, from (42) it follows that

$$B_k^T(G_k - zI_k)^{-1}B_kF_{n,k}^{-1} = \frac{a_k^2\gamma_{k,k}}{\det(F_{n,k})}\begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n-k} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \frac{\det(G_{n-k}^{(k)} - zI_{n-k})}{\det(F_{n,k})}B_k^T(G_k - zI_k)^{-1}B_k(G_{n-k}^{(k)} - zI_{n-k})^{-1}\,. \tag{46}$$

Consequently, in order to establish the uniform boundeness for the blocks $F_{n,k}^{-1}$ it is sufficient to prove it for the sequence

$$\left\{\frac{\det(G_{n-k}^{(k)} - zI_{n-k})}{\det(F_{n,k})}\right\}_{n\geq k+n_0}.$$

Developing $\det(F_{n,k})$ along its first row and taking into account its tridiagonal structure, we obtain that

$$\begin{aligned}\det(F_{n,k}) &= \det\begin{pmatrix} b_k - z - a_k^2\gamma_{k,k} & a_{k+1} & \dots & 0 & 0 \\ a_{k+1} & b_{k+1} - z & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & b_{n-2} - z & a_{n-1} \\ 0 & 0 & \dots & a_{n-1} & b_{n-1} - z \end{pmatrix} \\ &= \det(G_{n-k}^{(k)} - zI_{n-k}) - a_k^2\gamma_{k,k}\det(G_{n-k-1}^{(k+1)} - zI_{n-k-1}) \\ &= (-1)^{n-k}P_{n-k}^{(k)}(z) - a_k^2\gamma_{k,k}(-1)^{n-k-1}P_{n-k-1}^{(k+1)}(z),\end{aligned}$$

where $\gamma_{k,k} = -\frac{P_{k-1}(z)}{P_k(z)}$. Then, sequence

$$\frac{\det(G_{n-k}^{(k)} - zI_{n-k})}{\det(F_{n,k})} = \frac{P_{n-k}^{(k)}(z)}{P_{n-k}^{(k)}(z) - a_k^2\frac{P_{k-1}(z)}{P_k(z)}P_{n-k-1}^{(k+1)}(z)} = \frac{P_k(z)}{P_k(z) - a_k^2P_{k-1}(z)f_{n-k}^{(k)}(z)}$$

tends (as $n \to \infty$) to

$$\frac{P_k(z)}{P_k(z) - a_k^2P_{k-1}(z)f^{(k)}(z)} \neq \infty$$

(see (40)).

Hence, we have established that norms of finite-dimentional resolvents are bounded for each $z \in \mathbf{C} \setminus \left[\sigma_p(G) \cup \bigcap_{k=0}^{\infty} conv(\sigma(H^{(k)}))\right]$ fixed. In order to prove convergence, it remains to verify it on a dense subspace of l^2. We follow an idea similar to the one we have already used in the proof of Lemma 4.

Set $v_j = (G - zI)e_j\,,\ j = 0, 1, \ldots$; we need to know that $span\{v_j,\ j = 0, 1, \ldots\}$ is dense in ℓ^2. Although now G is, generally speaking, not selfadjoint, it verifies $g_{ij} = g_{ji}$ (see (1)), that is sufficient to establish that the only vector $x \in D(G)$ orthogonal to $span\{v_j,\ j = 0, 1, \ldots\}$ is $x = 0$.

Pointwise convergence in vectors v_i is immediate. Also the uniform convergence is obtained as in the proof of Lemma 4. ∎

Remark 5 Previous result may be of independent interest, since it gives us the structure of the matrix expression for $\mathcal{R}^{(G)}(z)$. In fact, it is sufficient to use block-wise convergence of matrix in (34).

Corollary 6

$$f_n(z) \underset{n}{\Longrightarrow} f(z) = - < \mathcal{R}^{(G)}(z)e_0, e_0 > \tag{47}$$

on each compact subset of $\mathbf{C} \setminus \left[\sigma_p(G) \cup \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)}))\right]$.

Proof: It is sufficient take into account that

$$|f_n(z) - f(z)| = | < \mathcal{R}_n^{(G)}(z)e_0 - \mathcal{R}^{(G)}(z)e_0, e_0 > | \leq \|\mathcal{R}_n^{(G)}(z)e_0 - \mathcal{R}^{(G)}(z)e_0\|.$$

■

Finally, we establish the relation between poles of $f(z)$, poles of its approximants and eigenvalues of G. We use the following result (see [4, Lemma 4]):

Lemma 9 *Under assumptions of Theorem 3,*

$$(\sigma_p(G^{(m)}) \cap \sigma_p(G^{(m+1)})) \setminus \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)})) = \emptyset$$

for all $m \geq 0$.

In the sequel, given a set $U \subset \mathbf{C}$, we denote by $\kappa(P_n, U)$ the number of zeros (taking into account multiplicities) of P_n in U.

Theorem 4 $f \in \mathcal{M}(\mathbf{C} \setminus \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)})))$ *(meromorphic in* $\mathbf{C} \setminus \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)}))$*), each point of* $\sigma_p(G) \setminus \bigcap_{k=0}^{\infty} \operatorname{conv}(\sigma(H^{(k)}))$ *is a pole of* f*. Moreover,*

$$\kappa(\zeta) = m(\zeta\,,\, G) \tag{48}$$

for each $\zeta \in \sigma_p(G)$*, where* $\kappa(\zeta)$ *denotes the order of* ζ *as a pole of* f *and* $m(\zeta\,,\, G)$ *denotes the algebraic multiplicity of* ζ *as an eigenvalue of* G.

Proof: Fix $\zeta \in \sigma_p(G) \setminus \bigcap_{k=0}^{\infty} conv(\sigma(H^{(k)}))$. It is known that it must be an isolated point of $\sigma_p(G)$, so there exists $\delta > 0$ such that

$$U(\zeta\,,\, \delta) \stackrel{\text{def}}{=} \{z : 0 < |z - \zeta| \leq \delta\} \subset \rho(G) \setminus \bigcap_{k=0}^{\infty} conv(\sigma(H^{(k)})).$$

Moreover, from Lemma 9 we know that $\zeta \notin \sigma_p(G^{(1)})$, so we may assume δ sufficiently small in order to verify $\overline{U(\zeta\,,\, \delta)} \cap \sigma(G^{(1)}) = \emptyset$. Hence, $P_{n-1}^{(1)}(z) \neq 0\,,\, \forall z \in U(\zeta\,,\, \delta)\,,\, \forall n \geq n_1(U(\zeta\,,\, \delta))$ (on the contrary, an accumulation point of zeros of $\{P_n^{(1)}\}_{n\in\mathbf{N}}$ in $\overline{U(\zeta\,,\, \delta)}$ should exist, which contradicts Lemma 8).

If we take

$$\Gamma \stackrel{\text{def}}{=} \{z : |z - \zeta| = \delta\},$$

from the uniform convergence of f_n on Γ (Corollary 6) it follows

$$\lim_n \frac{1}{2\pi i}\int_\Gamma \frac{f_n'(z)}{f_n(z)}\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz\,.$$

Using the argument principle and taking into account that integrals in the left-hand side are integers, we conclude that

$$\kappa(P_n, U(\zeta\,,\,\delta)) = \kappa(\zeta)\,,\,\forall n \geq n_2(U(\zeta\,,\,\delta)). \tag{49}$$

On the other hand, using Lemma 8 we find an $n_2 = n_2(\Gamma)$ such that $\Gamma \subset \rho(G_n)\,,\,\forall n \geq n_2$. Hence, we can define linear operators in ℓ^2

$$P(\zeta\,,\,\Gamma) = -\frac{1}{2\pi i}\int_\Gamma \mathcal{R}^{(G)}(z)dz\,,\quad P_n(\zeta\,,\,\Gamma) = -\frac{1}{2\pi i}\int_\Gamma \mathcal{R}_n^{(G)}(z)dz\,,\,n\geq n_2.$$

We have already established the uniform convergence of $\mathcal{R}_n^{(G)}(z)$ to $\mathcal{R}^{(G)}(z)$ on compact sets, and this means that

$$P_n(\zeta\,,\,\Gamma) \longrightarrow P(\zeta\,,\,\Gamma). \tag{50}$$

It is known (see [11], Theorem 6.17, page 178, and consequences) that, under our assumptions, one has that

$$\dim P(\zeta\,,\,\Gamma)\ell^2 = m(\zeta\,,\,G)\,,\quad \dim P_n(\zeta\,,\,\Gamma)\ell^2 = \kappa(P_n, U(\zeta\,,\,\delta)) = \kappa(\zeta). \tag{51}$$

Moreover, ζ is a possible pole of $\mathcal{R}^{(G)}(z)$ whose order is at most $m(\zeta\,,\,G)$. Then,

$$\mathcal{R}^{(G)}(z) = \sum_{\nu=-m(\zeta\,,\,G)}^{\infty} (z-\zeta)^\nu B_\nu \ \ (B_\nu \text{ bounded operators}),$$

where B_ν might be 0 for some $\nu \geq -m(\zeta\,,\,G)$ (see also [9], Theorem 1.3, page 26 and Theorem 2.2, page 326). From this and (47), because of Corollary 6 we have that

$$f(z) = -\sum_{\nu=-m(\zeta\,,\,G)}^{\infty} (z-\zeta)^\nu < B_\nu e_0\,,\,e_0 > .$$

But the order of ζ as pole of f is $\kappa(\zeta)$, thus

$$\kappa(\zeta) \leq m(\zeta\,,\,G).$$

Now, it remains to use the following Lemma, that we cite for completeness of the reading:

Lemma 10 [11, p. 438]: *Let* $\{P_n\}$, $n = 1, 2, \ldots,$ *be a sequence of projections in a Banach space* $\mathbf{X}$ *such that* $P_n \longrightarrow P \in \mathcal{B}(\mathbf{X})$. *Then* P *is also a projection. Suppose further that* $\dim P_n \leq \dim P < \infty$ *for all* n. *Then*

$$\dim P_n = \dim P$$

for sufficiently large n.

Taking into account (50) and (51) this gives us (48). Theorem is proved. ■

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